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SINGLY GENERATED HOMOGENEOUS F-ALGEBRAS

BY
RONN CARPENTER

Abstract. With each point m in the spectrum of a singly generated F-algebra we associate an algebra A_m of germs of functions. It is shown that if A_m is isomorphic to the algebra of germs of analytic functions of a single complex variable, then the spectrum of A contains an analytic disc about m. The algebra A is called homogeneous if the algebras A_m are all isomorphic. If A is homogeneous and none of the algebras A_m have zero divisors, we show that A is the direct sum of its radical and either an algebra of analytic functions or countably many copies of the complex numbers. If A is a uniform algebra which is homogeneous, then it is shown that A is either the algebra of analytic functions on an open subset of the complex numbers or the algebra of all continuous functions on its spectrum.

1. **Introduction.** Let A be a singly generated F-algebra with unit. With each point m in the spectrum of A we associate an algebra A_m of germs of functions. In §3 we prove that if the algebra A_m is isomorphic to the algebra of germs of analytic functions in one variable, then the point m lies in an analytic disc in the spectrum of A.

In case A is the algebra Hol (Ω) of analytic functions on an open polynomially convex subset Ω of the complex plane, then the spectrum of A is Ω and for a point m in Ω the algebra A_m is the algebra of germs of analytic functions at the point m of Ω. In this case for any two points m and n in Ω there is a natural isomorphism of A_m onto A_n induced by translation. For $A=\text{Hol}(\Omega)$ we also have that none of the algebras A_m contain algebraic divisors of zero (see [9, p. 67]). In §4 we define a singly generated F-algebra to be homogeneous if for any two points m and n in the spectrum of A there is an isomorphism of the algebra A_m onto the algebra A_n. It is shown that if A is a singly generated homogeneous F-algebra with unit and if none of the algebras A_m contain algebraic divisors of zero, then A is essentially an algebra of analytic functions in the sense that either $A=R(A)\oplus \text{Hol}(D)$ where $R(A)$ is the radical of A and D is an open polynomially convex subset of the plane, or $A=R(A)\oplus \sum C_i$ where C_i is a copy of the complex numbers and the sum is at most countable.

In §5 we specialize to uniform algebras and drop the restriction that the algebras A_m have no zero divisors. A complete characterization of singly generated uniform homogeneous F-algebras is obtained. Namely, if A is a singly generated uniform

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homogeneous F-algebra with unit, then either $A = \mathrm{Hol}(D)$ where D is an open polynomially convex subset of the plane, or A is the algebra of all continuous functions on its spectrum. An example is given to show how this characterization may fail for nonuniform algebras.

2. **Preliminaries.** In this paper all algebras are assumed to be commutative and contain units. An F-algebra A is a complete topological algebra over the complex numbers in which the topology is given by a countable family

$$\{\|\cdot\|_n : n = 1, 2, \ldots\}$$

of algebraic seminorms. It is easily seen that the seminorms can be assumed to be increasing; i.e., for each positive integer n and each a in A we can assume $\|a\|_n \leqq \|a\|_{n+1}$. For each seminorm $\|\cdot\|_n$ we can obtain a Banach algebra B_n by setting B_n equal to the completion of the quotient algebra $A/(\ker \|\cdot\|_n)$ with respect to the norm induced on $A/(\ker \|\cdot\|_n)$ by $\|\cdot\|_n$. The symbol π_n will denote the natural projection of the algebra A into the algebra B_n. The algebra A is the inverse limit of the Banach algebras B_n. The spectrum of A, denoted by Spec A, is the space of all continuous homomorphisms of A onto the complex numbers with the Gelfand topology. For each positive integer n, Spec B_n is a compact Hausdorff space which is embedded homeomorphically in Spec A and Spec $A = \bigcup_{n=1}^{\infty} \mathrm{Spec}\, B_n$. For any compact subset K of Spec A there is an integer n such that K is contained in Spec B_n. Hence Spec A is a σ-compact, hemicompact, Hausdorff space.

For an element a of A we denote by $a^\wedge$ the Gelfand transform of a and by $A^\wedge$ the algebra of all Gelfand transforms of elements of A. We can define seminorms on the algebra $A^\wedge$ by $|a^\wedge|_n = \max\{|a^\wedge(m)| : m \in \mathrm{Spec}\, B_n\}$ for each a in A. Since every compact subset of Spec A is contained in some Spec B_n and each Spec B_n is compact, the topology on $A^\wedge$ defined by these seminorms is the compact open topology.

In this paper the symbol $\mathbf{C}$ will stand for the complex plane, and for a subset S of $\mathbf{C}$ int S will denote the interior of S with respect to $\mathbf{C}$. The symbol $\mathcal{O}$ will denote the algebra of germs of analytic functions at the origin of $\mathbf{C}$ (see [9, p. 66]). The symbol A will always denote a singly generated F-algebra with unit.

3. **Analytic discs.** Let A be a singly generated F-algebra with unit and fix a generator x for A. We fix an increasing sequence $\{\|\cdot\|_n\}$ of algebraic seminorms which determine the topology of A and the corresponding sequence $\{B_n\}$ of Banach algebras. We identify Spec B_n with its homeomorphic image in Spec A and denote this subset of Spec A by M_n.

For each open subset U of Spec A we let $A(U)$ denote the completion of the algebra $A^\wedge | U$ with respect to the seminorms defined by

$$\|f\|_n^U = \sup\{|f(m)| : m \in U \cap M_n\}, \qquad n = 1, 2, \ldots.$$

Here $A^\frown|U$ denotes the algebra of Gelfand transforms of elements of A restricted to U. We note that if the closure of U is compact, then U is contained in M_n for some integer n. In this case $A(U)$ is a uniform Banach algebra contained in the algebra of all continuous functions on U. If U is not contained in any M_n, then $A(U)$ is an F-algebra. In the general case we do not know whether the elements of $A(U)$ are continuous on U with respect to the relative Gelfand topology on U. However, for each positive integer n the restrictions of the functions in $A(U)$ to $U \cap M_n$ are continuous on $U \cap M_n$ with respect to the relative Gelfand topology.

For each m in Spec A, A_m denotes the algebraic direct limit of the algebras $A(U)$ where the limit is taken over all open sets U which contain m directed by inclusion. If f is an element of $A(U)$ for some open set U containing m, then $\gamma_m(f)$ will denote the equivalence class of f in A_m.

The following lemma appeared in [13]. Consequently we will merely sketch a proof and refer the reader to [13] for the details of the proof.

LEMMA 3.1. *If m is a point of* Spec A *and m is isolated in each M_n which contains it, then m is isolated in* Spec A.

Proof. Suppose m is isolated in each M_n which contains it. We use Šilov's idempotent theorem on the Banach algebras B_n and the fact that the only idempotent in the radical of a Banach algebra is zero to obtain an idempotent e in A such that $e^\frown(m)=1$ and $e^\frown=0$ elsewhere on Spec A.

For the next lemma we fix a point m in Spec A and assume there is an isomorphism φ of A_m onto $\mathcal{O}$, the algebra of germs of analytic functions at the origin of C. Let U be an open set containing m. For each f in $A(U)$ choose a sequence $\{b_i^f\}_i$ of complex numbers such that $\varphi\gamma_m(f)=\sum_{i=0}^{\infty} b_i^f z^i$ ($\mathcal{O}$ is the algebra of all power series in z which have a positive radius of convergence). Recall that $A(U)$ is an F-algebra.

LEMMA 3.2. *For each positive integer i the functional $f \to b_i^f$ is continous.*

Proof. The linearity of $f \to b_i^f$ is clear. Let f be an element of $A(U)$. Since $\gamma_m(f-f(m))$ is in the ideal of A_m consisting of all equivalence classes with representing functions which are zero at m, $\varphi\gamma_m(f-f(m))$ is contained in the unique maximal ideal of $\mathcal{O}$. Therefore, the constant term in the power series $\varphi\gamma_m(f-f(m))$ is zero. Since $\varphi\gamma_m(f-f(m))=\varphi\gamma_m(f)-\varphi\gamma_m(f(m))=\sum_{i=0}^{\infty} b_i^f z^i-f(m)$, we have $b_0^f=f(m)$. From this it is clear that $f \to b_0^f$ is a continuous functional on $A(U)$.

We prove that the mappings $f \to b_i^f$, $i=1, 2, \ldots$ are continuous by induction on i. Define elements σ_n^f of $\mathcal{O}$ by $\sigma_n^f=\sum_{i=n}^{\infty} b_i^f z^{i-n}$ for each f in $A(U)$ and $n=1, 2, \ldots$. Choose open sets V, $U_{i,f}$, $i=1, 2, \ldots$ containing m and functions g and h_i^f in $A(V)$ and $A(U_{i,f})$ respectively such that $\varphi\gamma_m(g)=z$ and $\varphi\gamma_m(h_i^f)=\sigma_i^f$, $i=1, 2, \ldots$. Since $\varphi\gamma_m(g)$ is an algebraic generator for the unique maximal ideal in $\mathcal{O}$, $\gamma_m(g)$ must generate the unique maximal ideal in A_m. Recall that x denotes a fixed generator for the algebra A. The maximal ideal in A_m contains $\gamma_m(x^\frown-x^\frown(m))$ so there is an open set W containing m and a function h in $A(W)$ such that $x^\frown-x^\frown(m)=hg$ on W.

Since x generates A, $x^\wedge$ is one-to-one on Spec A. Hence m is the only zero of g on W.

We have assumed m is not isolated in Spec A. Lemma 3.1 implies that there is an integer n_0 such that m is not isolated in M_{n_0}. Since M_{n_0} is homeomorphic to a subset of C it is first countable and we can choose a sequence $\{m_i\}$ from $(M_{n_0}-\{m\}) \cap W \cap U$ such that $\lim_i m_i = m$.

For each f in $A(U)$ define $\Phi_i^1(f) = [f(m_i)-f(m)]/g(m_i)$. The map $f \to \Phi_i^1(f)$ is a continuous linear functional on $A(U)$ for each integer i. Consider the previously defined function h_1^f. We have $\varphi\gamma_m(h_1^f) = \sum_{i=1}^{\infty} b_i^f z^{i-1}$. Hence $\varphi\gamma_m(gh_1^f) = \sum_{i=1}^{\infty} b_i^f z^i = \varphi\gamma_m(f-f(m))$. Therefore, there is an open set V_1 containing m such that $gh_1^f = f-f(m)$ on V_1. Since $V_1 \cap M_{n_0}$ is open in M_{n_0} and $\lim_i m_i = m$, for i sufficiently large we have $h_1^f(m_i) = [f(m_i)-f(m)]/g(m_i) = \Phi_i^1(f)$. The restriction of any function in $A(U_{i,f})$ to $M_{n_0} \cap U_{i,f}$ is continuous. Hence, $\lim_i h_1^f(m_i) = h_1^f(m)$ and $\lim_i \Phi_i^1(f) = h_1^f(m)$. Thus, the sequence $\{\Phi_i^1\}$ is a sequence of continuous functionals on $A(U)$ and for any f in $A(U)$ we have $\lim_i \Phi_i^1(f) = h_1^f(m)$. Since $A(U)$ is an F-space the uniform boundedness principle implies that $f \to h_1^f(m)$ is a continuous functional on $A(U)$ (see [7, p. 54]). An argument similar to the one used to show $b_0^f = f(m)$ will show that $b_1^f = h_1^f(m)$. Therefore $f \to b_1^f$ is continuous.

Define sequences $\{\Phi_i^j\}_{i=1}^{\infty}$ $(j=1, 2, \dots)$ of functionals on $A(U)$ inductively by

$$\Phi_i^1(f) = [f(m_i)-f(m)]/g(m_i) \quad \text{and} \quad \Phi_i^{j+1}(f) = [\Phi_i^j(f)-h_j^f(m)]/g(m_i).$$

Fix k and suppose the functionals $f \to b_k^f = h_k^f(m)$ and $f \to \Phi_i^k(f)$, $i=1, 2, \dots$ are continuous, and that for large i we have $\Phi_i^k(f) = h_k^f(m_i)$. Then $f \to \Phi_i^{k+1}(f)$ is continuous and for i sufficiently large we have $\Phi_i^{k+1}(f) = [h_k^f(m_i)-h_k^f(m)]/g(m_i)$. Now $\varphi\gamma_m(gh_{k+1}^f) = \varphi\gamma_m(h_k^f - h_k^f(m))$, so there is an open set V_2 such that $m \in V_2$ and $gh_{k+1}^f = h_k^f - h_k^f(m)$ on V_2. Since $\lim_i m_i = m$ and $g(m_i) \neq 0$ we have $h_{k+1}^f(m_i) = [h_k^f(m_i)-h_k^f(m)]/g(m_i)$. Thus for large i we have $\Phi_i^{k+1}(f) = h_{k+1}^f(m_i)$. Since the restriction of h_{k+1}^f to $M_{n_0} \cap U_{k+1,f}$ is continuous $\lim_i \Phi_i^{k+1}(f) = \lim_i h_{k+1}^f(m_i) = h_{k+1}^f(m)$. An application of the uniform boundedness principle to the sequence of functionals $\{\Phi_i^{k+1}\}_i$ yields $f \to h_{k+1}^f(m) = b_{k+1}^f$ is a continuous functional on $A(U)$. Mathematical induction now gives the desired conclusion.

For the next lemma we fix an element m of Spec A. Let φ be an isomorphism of A_m onto $\mathcal{O}$. Let U be an open set containing m. Define a homomorphism ψ: $A(U) \to \mathcal{O}$ by $\psi(f) = \varphi\gamma_m(f)$ for each f in $A(U)$.

Lemma 3.3. *There is a positive number δ such that ψ maps $A(U)$ into the subalgebra of $\mathcal{O}$ consisting of all power series which have a radius of convergence greater than or equal to δ.*

Proof. For each positive integer n set $F_n = \{f \in A(U) : \sup_i |b_i^f|^{1/i} \leq n$, where $\psi(f) = \sum_{i=0}^{\infty} b_i^f z^i\}$. We can write F_n as $F_n = \bigcap_i \{f \in A(U) : |b_i^f| \leq n^i\}$. Lemma 3.2 implies that each of the sets in this intersection is closed. Therefore, F_n is closed. Note that $A(U) = \bigcup_{n=1}^{\infty} F_n$. Since $A(U)$ is an F-space and F_n is closed for each n, the Baire category theorem yields the existence of an integer p such that the interior of F_p is nonempty. Choose an element f_0 of int F_p. Note that zero belongs int $(F_p - f_0)$. Let f be an element of $A(U)$. Since int $(F_p - f_0)$ is an open set containing zero, there is a constant $\lambda > 0$ such that $\lambda f \in \text{int}\,(F_p - f_0)$. Set $\lambda f = g - f_0$ where g is an element of int F_p. Then $\psi(f) = \lambda^{-1}\psi(\lambda f) = \lambda^{-1}[\psi(g) - \psi(f_0)]$. Let r be the radius of convergence of the power series $\psi(f_0)$. Set $\delta = \min(r, p^{-1})$. Since $\psi(g)$ has a radius of convergence at least as large as p^{-1}, the radius of convergence of $\psi(f)$ must be greater than or equal to δ. Therefore ψ maps $A(U)$ into the algebra of all power series having radius of convergence greater than or equal to δ.

Let A be an F-algebra and m be a point of Spec A. The point m is contained in an analytic disc if there is a homeomorphism ψ of the open unit disc in $\mathbf{C}$ into Spec A satisfying; (1) $\psi(0) = m$, and (2) for any a in A the function $a^\wedge \psi$ is analytic in the unit disc.

THEOREM 3.4. *If there is an isomorphism of A_m onto $\mathcal{O}$, then m is contained in an analytic disc.*

Proof. Suppose φ is an isomorphism of A_m onto $\mathcal{O}$. Let U be an open set containing m and g be an element of $A(U)$ such that $\varphi\gamma_m(g) = z$. Lemma 3.3 implies there is a closed disc Δ centered at the origin of $\mathbf{C}$ such that $\varphi\gamma_m[A(U)]$ is contained in the algebra B of all complex-valued functions on Δ which are continuous on Δ and analytic in int Δ. Define a homomorphism $\psi_1 \colon A \to A(U)$ by $\psi_1(a) = a^\wedge | U$ where a is any element of A and $a^\wedge | U$ denotes the restriction of $a^\wedge$ to U. Define a homomorphism $\psi_2 \colon A(U) \to B$ by $\psi_2(f) = \varphi\gamma_m(f)$ for any f in $A(U)$. Here we have identified the power series $\varphi\gamma_m(f)$ with the function to which it converges on Δ. Let h be a homomorphism of B onto $\mathbf{C}$. It is shown in [1] that every homomorphism of a singly generated F-algebra onto $\mathbf{C}$ is continuous. Since $A(U)$ is singly generated, by $\psi_1(x)$, we have that $h\psi_2$ is a continuous homomorphism of $A(U)$ onto $\mathbf{C}$.

If B is normed with the supremum norm it becomes a Banach algebra whose spectrum is Δ. Let h_1 and h_2 be homomorphisms of B onto $\mathbf{C}$ corresponding to distinct points of Δ. Since $\psi_2(g) = z$ we have $h_1\psi_2(g) \neq h_2\psi_2(g)$. Since $h_1\psi_2$ and $h_2\psi_2$ are continuous on $A(U)$ and $\psi_1(A)$ is dense in $A(U)$ there is an a_0 in A such that $h_1\psi_2\psi_1(a_0) \neq h_2\psi_2\psi_1(a_0)$.

Set $\psi = \psi_2\psi_1$. Referring to [1] we have that since A is a singly generated F-algebra every homomorphism of A onto $\mathbf{C}$ is continuous. Therefore, the adjoint map ψ^* of ψ takes Δ into Spec A. Since $\psi(A)$ separates the points of Δ, we have that ψ^* is one-to-one. Clearly, ψ^* is continuous. Hence, ψ^* is a one-to-one continuous map of the compact space Δ into the Hausdorff space Spec A. Therefore, ψ^* is a homeomorphism of Δ into Spec A. Moreover, $\psi^*(0)$ is m, since for each a in A, we have $[\psi^*(0)](a) = [\psi(a)](0) = a^\wedge(m)$.

If a is an element of A, then since $\psi(a)$ is analytic on int Δ, $a^\wedge \circ \psi^*$ must be analytic on int Δ. Therefore the point m is contained in an analytic disc.

4. Homogeneity. Let A be a singly generated F-algebra with unit. As in §3, we fix a sequence $\{B_n\}$ of Banach algebras such that $A = \lim \operatorname{inv} B_n$. We identify $\operatorname{Spec} B_n$ with its homeomorphic image M_n in $\operatorname{Spec} A$. Fix a generator x for A. The map $m \to x^\wedge(m)$ is a one-to-one continuous map of $\operatorname{Spec} A$ onto a subset D of C. In general this map is not a homeomorphism nor is D open in C (see [4]). It may even occur that this map is a homeomorphism for some generators and not for others. However, for each positive integer n the restriction of $m \to x^\wedge(m)$ to M_n is a homeomorphism of M_n onto a compact subset D_n of C.

Recall that with each point m of $\operatorname{Spec} A$ we have associated the algebra A_m of germs of functions which are locally approximable at m by functions in $A^\wedge$.

DEFINITION. The algebra A will be called homogeneous provided that for each pair m, n of points of $\operatorname{Spec} A$, there is an algebra isomorphism of A_m onto A_n.

If m is isolated in $\operatorname{Spec} A$, then $A_m = C$. If m is not isolated in $\operatorname{Spec} A$, then the function $x^\wedge$ is nonconstant on every open set containing m. Hence, if m is isolated in $\operatorname{Spec} A$ and n is a nonisolated point of $\operatorname{Spec} A$ the corresponding algebras A_m and A_n are not isomorphic. Therefore, the spectrum of a singly generated homogeneous algebra contains an isolated point if, and only if, every point of the spectrum is isolated.

The next lemma characterizes algebras whose spectra contain only isolated points.

LEMMA 4.1. *If the spectrum of A contains only isolated points, then there is a closed subalgebra A_0 of A such that*

(1) *A_0 is topologically isomorphic to a direct sum of countably many copies of C, and*

(2) *$A = A_0 \oplus R(A)$ where $R(A)$ is the radical of A.*

Proof. Suppose $\operatorname{Spec} A$ contains only isolated points. Since each M_n is compact it contains at most finitely many points; hence, $\operatorname{Spec} A$ is at most countable. For each point m_i, $i = 1, 2, \ldots$ in $\operatorname{Spec} A$ we construct, as in the proof of Lemma 3.1, an idempotent e_i such that $e_i^\wedge(m_j) = \delta_{ij}$.

Recall that we have fixed an increasing sequence $\{\|\cdot\|_n\}$ of algebraic seminorms which determine the topology of A. Since zero is the only idempotent in the radical of a Banach algebra, we have that for each positive integer n there is an integer i_n such that $\|e_j\|_n = 0$ for $j \geqq i_n$. It follows that for any function f which maps $\operatorname{Spec} A$ into C, the sequence $\{\sum_{i=1}^n f(m_i)e_i\}_n$ is a Cauchy sequence with respect to each of the seminorms $\|\cdot\|_j$. Therefore $\sum_{i=1}^\infty f(m_i)e_i$ converges to an element of A.

Let A_0 be the subalgebra of A consisting of all elements of the form $\sum_{i=1}^\infty f(m_i)e_i$ where f is any function which maps $\operatorname{Spec} A$ into C. Suppose that $\{b_i\}$ is a sequence of elements from A_0 and that the sequence $\{b_i\}$ converges to an element b of A. Fix a seminorm $\|\cdot\|_n$. Then

$$\left\| b-\sum_{i=1}^{\infty} b\hat{\ }(m_i)e_i \right\|_n \leqq \|b-b_j\|_n+\left\| b_j-\sum_{i=1}^{\infty} b\hat{\ }(m_i)e_i \right\|_n$$
$$= \|b-b_j\|_n+\left\| \sum_{i=1}^{\infty} b_j\hat{\ }(m_i)e_i-\sum_{i=1}^{\infty} b\hat{\ }(m_i)e_i \right\|_n$$
$$\leqq \|b-b_j\|_n+\sum_{i=1}^{k} |b_j\hat{\ }(m_i)-b\hat{\ }(m_i)|\cdot\|e_i\|_n$$

where k is an integer such that for $i \geqq k$ we have $\|e_i\|_n=0$. Now since $\{b_j\}$ converges to b we see that $\|b-\sum_{i=1}^{\infty} b\hat{\ }(m_i)e_i\|_n=0$. Since n was arbitrary we have $b=\sum_{i=1}^{\infty} b\hat{\ }(m_i)e_i$. Therefore, b is an element of A_0. This shows that A_0 is a closed subalgebra of A. We note that $A_0 \cap R(A)=0$ and that A_0 is topologically isomorphic to the direct sum of countably many copies of C. For any a in A we have

$$a = \left(a-\sum_{i=1}^{\infty} a\hat{\ }(m_i)e_i\right)+\sum_{i=1}^{\infty} a\hat{\ }(m_i)e_i,$$

where $a-\sum_{i=1}^{\infty} a\hat{\ }(m_i)e_i$ is in $R(A)$ and $\sum_{i=1}^{\infty} a\hat{\ }(m_i)e_i$ is in A_0. Therefore

$$A = A_0 \oplus R(A).$$

Recall that we defined subsets D_n of C at the beginning of this section by $D_n=x\hat{\ }(M_n)$.

LEMMA 4.2. *If* int $D_n=\varnothing$ *for each positive integer n and m is a nonisolated point of Spec A, then A_m contains algebraic divisors of zero.*

Proof. Suppose int $D_n=\varnothing$ for every n and let m be a nonisolated point of Spec A. Lemma 3.1 implies that there is an integer k such that m is not isolated in M_k. The set M_k is homeomorphic to a subset of C. Hence, M_k is first countable and we can choose a sequence m_i of distinct points from $M_k-\{m\}$ such that $\lim_i m_i=m$. The sequence $\{x\hat{\ }(m_i)\}$ is a sequence of distinct points in C and $\lim_i x\hat{\ }(m_i)=x\hat{\ }(m)$. Choose open sets U_i in C such that $x\hat{\ }(m_i)\in U_i$ and $U_i \cap U_j=\varnothing$ for $i\neq j$. Let h_i be a continuous function on C such that $h_i(x\hat{\ }(m_i))=1$, $\sup\{|h_i(z)| : z\in C\}=1$, and h_i has its support in U_i. Set $f=\sum_{i=1}^{\infty} 2^{-i}h_{2i}$ and $g=\sum_{i=1}^{\infty} 2^{-i}h_{2i+1}$. The functions f and g are continuous on C and satisfy $fg=0$, $f(x\hat{\ }(m_{2i}))\neq 0$, $g(x\hat{\ }(m_{2i+1}))\neq 0$.

For each integer n the set D_n is a compact polynomially convex subset of C and int $D_n=\varnothing$. (A compact subset K of C is said to be polynomially convex if for any complex number z_0 not in K there is a polynomial p such that $|p(z_0)| > \sup\{|p(z)| : z\in K\}$.) It follows from Mergelyan's theorem on polynomial approximation that any continuous function on D_n can be uniformly approximated by polynomials (see [11]). Therefore, the functions $f\circ x\hat{\ }$ and $g\circ x\hat{\ }$ are in the algebra $A(U)$ for $U=$Spec A.

If U is an open subset of Spec A which contains m, then $U\cap M_k$ is open in M_k

and contains m. Hence, U contains all but finitely many of the points m_i. Thus both $f \circ x^\wedge$ and $g \circ x^\wedge$ assume nonzero values on U. Therefore $\gamma_m(f \circ x^\wedge) \neq 0$ and $\gamma_m(g \circ x^\wedge) \neq 0$. Since $\gamma_m(f \circ x^\wedge)\gamma_m(g \circ x^\wedge) = \gamma_m[(f \circ x^\wedge)(g \circ x^\wedge)] = \gamma_m(0)$, the algebra A_m has algebraic zero divisors.

LEMMA 4.3. *If there is an isomorphism φ of A_m onto $\mathcal{O}$, then there is an integer n such that $x^\wedge(m)$ is contained in* int (D_n).

Proof. We conclude from Theorem 3.4 that there is a homeomorphism ψ^* of a closed disc Δ centered at the origin of $\boldsymbol{C}$ into Spec A such that $\psi^*(0) = m$. Since $\psi^*(\Delta)$ is compact there is an integer n such that $\psi^*(\Delta)$ is contained in M_n. Since $x^\wedge$ maps M_n homeomorphically onto D_n, the composition $x^\wedge\psi^*$ maps int (Δ) homeomorphically into $\boldsymbol{C}$. The invariance of domain theorem implies $x^\wedge\psi^*(\text{int } \Delta)$ is an open subset of $\boldsymbol{C}$. Since $\psi^*(0) = m$ and $\psi^*(\Delta)$ is contained in M_n we have that $x^\wedge(m)$ is in int (D_n).

THEOREM 4.4. *Suppose that A is a singly generated homogeneous F-algebra and that for each m in* Spec A *the algebra A_m has no algebraic zero divisors. Then either*

(1) *$A = R(A) \oplus \sum C_i$ where $C_i = \boldsymbol{C}$ and the sum is countable, or*

(2) *D is open in $\boldsymbol{C}$ and $A = R(A) \oplus$* Hol (D).

Proof. If every point of Spec A is isolated, then Lemma 4.1 implies that $A = R(A) \oplus \sum C_i$ where $C_i = \boldsymbol{C}$ and the sum is countable.

Suppose Spec A contains a point m which is not isolated. Since A_m has no zero divisors Lemma 4.2 implies int $D_n \neq \varnothing$ for some integer n. Let m_0 be a point of Spec A such that $x^\wedge(m_0)$ is in int D_n.

Identify M_n and D_n by the homeomorphism $m \to x^\wedge(m)$. For any open set U containing m_0 and satisfying $U \subset M_n$ the algebra $A(U)$ is the completion of the polynomials with respect to the supremum norm on U. Since the Euclidean and Gelfand topologies agree on M_n, and int D_n is nonempty, the algebra A_{m_0} is the direct limit of a family of algebras $A(U)$ such that the open sets U form a base for the topology of $\boldsymbol{C}$ at m_0 and $A(U)$ is the completion of the polynomials with respect to the supremum norm on U. This is sufficient to guarantee that A_{m_0} is isomorphic to $\mathcal{O}$ the algebra of germs of analytic functions at the origin of $\boldsymbol{C}$.

Consider an arbitrary element m of Spec A. Since A is homogeneous, A_m is isomorphic to A_{m_0}, which is isomorphic to $\mathcal{O}$. An application of Lemma 4.3 yields that $x^\wedge(m)$ is in int D_j for some integer j. Therefore $D = \bigcup_{j=1}^{\infty} \text{int } D_j$. This last equality implies that D is open in $\boldsymbol{C}$. We denote by Hol (D) the algebra of all functions which are analytic on D.

The equality $D = \bigcup \text{int } D_j$ allows us to apply a standard construction using the Cauchy integral formula to obtain a topological isomorphism φ of Hol (D) onto a closed subalgebra A_0 of A. The isomorphism φ has the property that if f belongs to Hol (D), then $\varphi(f)^\wedge(m) = f(x^\wedge(m))$. The reader is referred to [1] for the details of this construction.

Let a be an element of A. There is a sequence $\{p_j\}$ of polynomials such that $\lim_j p_j(x)=a$. Identifying Spec A with D by means of $m \to x^\frown(m)$ we have $\{p_j(z)\}$ converges uniformly to $a^\frown(z)$ on each of the compact subsets D_k of D. Since $D=\bigcup \operatorname{int} D_k$, every compact subset of D is contained in some D_k. Therefore $\{p_j(z)\}$ converges to $a^\frown(z)$ with respect to the compact-open topology on $C(D)$, where $C(D)$ denotes the algebra of all complex-valued continuous functions on D. This implies that $a^\frown$ is in Hol (D).

For any element a of A we have $a=[a-\varphi(a^\frown)]+\varphi(a^\frown)$. Hence, $A=R(A)+A_0$. Moreover, since $A_0 \cap R(A)=\{0\}$, we have that A is the direct sum of its radical and the closed subalgebra A_0. Therefore $A=R(A) \oplus A_0=R(A) \oplus \text{Hol}\,(D)$.

5. **Uniform algebras.** An F-algebra B is called uniform if its topology is determined by a sequence $\{\|\cdot\|_n\}$ of seminorms such that $\|b^2\|_n=\|b\|_n^2$ for each element b of B and each positive integer n. If B is a uniform algebra then for any $b \in B$ and positive integer n we have $\|b\|_n=\sup\{|b^\frown(m)| : m \in \text{Spec } B_n\}$ where B_n is the completion of the algebra $A/\ker \|\cdot\|_n$. Thus the map $b \to b^\frown$ which takes an element b of B onto its Gelfand transform is an isometry of B onto a complete subalgebra of $C(\text{Spec } B)$ the algebra of all continuous functions on Spec B. Where seminorms $\{|\cdot|_n\}$ are defined on $C(\text{Spec } B)$ by $|f|_n=\sup\{|f(m)| : m \in \text{Spec } B_n\}$. Since every compact subset of Spec B is contained in some Spec B_n and each Spec B_n is compact the topology defined on $C(\text{Spec } B)$ by the seminorms $\{|\cdot|_n\}$ is the compact-open topology. Therefore, a uniform F-algebra is a complete, hence closed, subalgebra of the algebra of continuous functions on a hemicompact Hausdorff space. Conversely, a subalgebra of the algebra of all continuous functions on a hemicompact Hausdorff space which is complete with respect to the compact-open topology is a uniform F-algebra.

Let (X, τ) be a Hausdorff topological space and $\{X_n\}$ be a sequence of compact subsets of X. We denote by (X, δ) the set X with the weak topology δ generated by the sequence $\{X_n\}$. A set S in X is δ-closed if, and only if, $S \cap X_n$ is τ-compact for each positive integer n. For a more detailed discussion of this topology the reader is referred to [6].

LEMMA 5.1. *Let X be a completely regular Hausdorff space and $\{X_n\}$ be an ascending sequence of compact subsets of X such that $\bigcup X_n=X$. Then $C(X, \delta)$, the algebra of all continuous functions on (X, δ) where δ denotes the weak topology generated by the compact subsets X_n, is an F-algebra with respect to the seminorms $\{|\cdot|_n\}$ defined by $|f|_n=\sup\{|f(x)| : x \in X_n\}$ and the spectrum of $C(X, \delta)$ is (X, δ).*

Proof. A function f on X is continuous on (X, δ) if, and only if, each of the restrictions $f|X_n$ is continuous. It is clear from this that $C(X, \delta)$ is complete with respect to the seminorms $\{|\cdot|_n\}$.

It is easy to see that the spectrum of $C(X, \delta)$ and X can be identified as sets. It is also clear that the Gelfand topology on X is weaker than the δ-topology.

To see that the Gelfand topology is as strong as the δ-topology we must show that every closed subset of (X, δ) is closed with respect to the Gelfand topology on X. Let S be a closed subset of (X, δ) and p be a point in $X-S$. Since X is completely regular and $\{X_n\}$ is an ascending sequence it is possible to construct a function f in $C(X, \delta)$ such that $f(p)=1$ and f is identically zero on S. This implies that S is closed with respect to the Gelfand topology on X.

In the next theorem A is a singly generated F-algebra. We fix a sequence $\{B_n\}$ of Banach algebras such that $A=\lim\text{ inv } B_n$. We fix a generator x for A and define subsets D_n of C by $D_n=x^\frown(\text{Spec } B_n)$. We set $D=\bigcup D_n$.

THEOREM 5.2. *Suppose that A is a singly generated uniform F-algebra and that A is homogeneous. Then either*

(1) *D is open in C and $A=\text{Hol}(D)$, or*

(2) *$A=C(\text{Spec } A)$ the algebra of all continuous functions on* Spec A.

Proof. If int $D_n \neq \varnothing$ for some integer n, then it follows from the proof of Theorem 4.4 that D is open in C and $A=R(A)\oplus\text{Hol}(D)$. Since A is uniform $R(A)=\{0\}$. Hence $A=\text{Hol}(D)$.

Now suppose int $D_n=\varnothing$ for each integer n. For each integer n, the set D_n is a compact polynomially convex subset of C. Mergelyan's theorem on polynomial approximation implies that the uniform completion of the polynomials on D_n is $C(D_n)$ the algebra of all continuous functions on D_n. Recall that A is the inverse limit of the sequence $\{B_n\}$ of Banach algebras and that the spectrum M_n of B_n is homeomorphic to the subset D_n of C. Since A is a uniform algebra, each B_n is a uniform algebra. This implies that the map $b \to b^\frown$ where $b \in B_n$ is an isometry of B_n into $C(D_n)$, where the norm in $C(D_n)$ is the supremum norm, and we have identified M_n and D_n. Since the uniform closure of the polynomials in $C(D_n)$ is $C(D_n)$, we have $B_n=C(D_n)$.

We now have $A=\lim\text{ inv } C(D_n)=C(D, \delta)$ where δ denotes the weak topology on D generated by the compact subsets $\{D_n\}$. Since D is a subset of C, D is completely regular. Lemma 5.1 implies Spec $[C(D, \delta)]=(D, \delta)$. Since $A=C(D, \delta)$, we have Spec $A=(D, \delta)$ and $A=C(\text{Spec } A)$.

EXAMPLE 5.3. We give an example which shows that the characterization given by Theorem 5.2 cannot be extended to nonuniform F-algebras, and indicates the type of algebras one must deal with in attempting to characterize nonuniform homogeneous algebras.

Let $A=C^1(R)$ the algebra of all continuously differentiable functions on the real line. Define seminorms $\{\|\cdot\|_n\}$ on A by

$$\|f\|_n = \max\{|f(x)| : x \in [-n, n]\}+\max\{|f'(x)| : x \in [-n, n]\}.$$

A with these seminorms is an F-algebra. We list below some of the properties of A.

(1) A is singly generated by the function f defined by $f(x)=x$ for each $x \in R$.

(2) Spec $A=R$, which is locally compact and connected.

(3) A is homogeneous.

(4) A is not C(Spec A) nor is $f^{\wedge}$ (Spec A) open in C.

6. **Remarks.** The algebras A_m appear to depend on the choice of the Banach algebras B_n in the representation $A=\lim\text{inv } B_n$. To see that this is not the case we assume two representations $A=\lim\text{inv } B_n$ and $A=\lim\text{inv } C_n$. Let $|\cdot|_n$ and $\|\cdot\|_n$ be the seminorms on A corresponding to the Banach algebras B_n and C_n respectively. For each integer i, $|\cdot|_i$ is a continuous convex functional on A. Hence, there is an integer j and a constant K such that $|\cdot|_i \leqq K\|\cdot\|_j$. This implies that the spectrum of B_i is contained in the spectrum of C_j and this in turn is sufficient to guarantee that the algebras A_m are independent of the representation of A.

Many of the theorems obtained for singly generated F-algebras extend immediately to algebras which are singly rationally generated. However, the extension of Theorems 4.4 and 5.2 is complicated by the fact that there are compact subsets K of C with int $K=\varnothing$ such that the algebra of rational functions with poles off K is not uniformly dense in the algebra of all continuous functions on K (see [10]). It does, however, appear possible to extend Theorem 4.4 to singly rationally generated algebras by using a theorem of Vitushkin's on rational approximation. A slightly modified version of Theorem 5.2 should also be valid for singly rationally generated F-algebras.

Another possible topic for research is the generalization of these results to finitely generated algebras.

Bibliography

1. R. Arens, *Dense inverse limit rings*, Michigan Math. J. **5** (1958), 169–182. MR **21** #3780.

2. F. T. Birtel, *Singly-generated Liouville F-algebras*, Michigan Math. J. **11** (1964), 89–94. MR **28** #4376.

3. R. M. Brooks, *Boundaries for locally m-convex algebras*, Duke Math. J. **34** (1967), 103–116. MR **37** #762.

4. ———, *On the spectrum of finitely generated locally M-convex algebras*, Studia Math. **29** (1968), 143–150.

5. ———, *On singly-generated locally M-convex algebras* (preprint).

6. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR **33** #1824.

7. N. Dunford and J. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

8. A. Gleason, *Finitely generated ideals in Banach algebras*, J. Math. Mech. **13** (1964), 125–132. MR **28** #2458.

9. R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965. MR **31** #4927.

10. R. McKissick, *A nontrivial normal sup norm algebra*, Bull. Amer. Math. Soc. **69** (1963), 391–395. MR **26** #4166.

11. S. N. Mergelyan, *On the representation of functions by series of polynomials on closed sets*, Dokl. Akad. Nauk SSSR **78** (1951), 405–408; English transl., Amer. Math. Soc. Transl. (1) **3** (1962), 287–293. MR **13**, 23.

12. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. No. 11 (1952). MR 14, 482.

13. M. Rosenfeld, *Commutative F-algebras*, Pacific J. Math. **16** (1966), 159–166. MR **32** #8196.

University of Houston,
Houston, Texas 77004